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Influence of Friction on the Direct Cascade of the 2d Forced Turbulence.

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Abstract.
We discuss two possible scenario for the direct cascade in two dimensional turbulent systems in presence of friction which differ by the presence or not of enstrophy dissipation in the inviscid limit.They are distinguished by the existence or not of a constant enstrophy transfer and by the presence of leading anomalous scaling in the velocity three point functions. We also point out that the velocity statistics become gaussian in the approximation consisting in neglecting odd order correlations in front even order ones.

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Two-dimensional turbulence has very peculiar properties compared to 3d turbulence, see eg. ref.[1, 2] for references. As first pointed out by Kraichnan in a remarkable paper [3], these open the possibility for quite different scenario for the behaviors of turbulent flows in 2d and 3d: if energy and enstrophy density are injected at a scale L_i, with respective rate $\overline{\epsilon}$ and $\overline{\epsilon}_w \simeq \overline{\epsilon} L_i^{-2}$, the 2d turbulent systems should react such that the energy flows toward the large scales and the enstrophy towards the small scales. One usually refers to the infrared energy flow as the inverse cascade and to the ultraviolet enstrophy flow as the direct cascade. In the (IR) inverse cascade, scaling arguments lead to Kolmogorov's spectrum, with $E(k) \sim \overline{\epsilon}^{2/3}\, k^{-5/3}$ for the energy and $(\delta u) \sim (\overline{\epsilon} r)^{1/3}$ for the variation of the velocity on scale r. In the (UV) direct cascade, scaling arguments give Kraichnan's spectrum with $E(k) \sim \overline{\epsilon}_w^{2/3}\, k^{-3}$ for the energy and $(\delta u) \sim (\overline{\epsilon}_w r^3)^{1/3}$ for the velocity variation. The aim of this Letter is to discuss a few possible scenario for the influence of friction on the direct enstrophy cascade of 2d turbulence. The picture which emerges is that at small scales the velocity field has a smooth gaussian component, with amplitude diverging in the frictionless limit, supplemented by possible anomalous corrections whose characters depend whether there is entrophy dissipation or not in the inviscid limit. We also point out that friction provides a way to regularize amplitudes in the inviscid limit.

A few experimental facts. Two-dimensional turbulence has recently been observed in remarkable experiments providing new informations on these peculiar characteristics. See refs.[4, 5] for a precise description of the experimental set-up. The turbulent flow takes place in a square cell such that the injection length is $L_i \simeq 10\ cm$. The bottom of the cell induces friction with a coefficient $\tau \simeq 25\ s$. The dissipation length l_d is of order 1 mm and the UV friction length l_f of order 0.5 cm. ($\tau,\ l_d,\ l_f$ are defined below). The enstrophy transfer rate determined from the three point structure function has been evaluated in [5] as $\eta_w \simeq 0.4\ s^{-3}$. The direct cascade is observed at intermediate scales between 1 cm and 10 cm. Besides the absence of any experimentally significant deviations from Kraichnan's scaling, two remarkable facts have been observed: (i) the vorticity structure functions are almost constant on the inertial range with the two-point function of order 10 s^{-2}; and (ii) the vorticity probability distribution function is almost symmetric and almost, but not quite, gaussian.

A model system. As usual, to statistically model turbulent flows we consider the Navier-Stokes equation with a forcing term. Let $u^j(x,t)$ be the velocity field for an incompressible fluid, $\nabla\cdot u = 0$. The Navier-Stockes equation with friction reads:

$$\partial_t u^j + (u\cdot\nabla)u^j - \nu\nabla^2 u^j + \frac{1}{\tau}\, u^j = -\nabla^j p + f^j \tag{1}$$

with p the pressure and $f(x,t)$ the external force such that $\nabla\cdot f = 0$. The pressure is linked to the velocity by the incompressibility condition: $\nabla^2 p = -\nabla^i\nabla^j u^i u^j$. The friction term $\frac{1}{\tau}\, u^j$ is introduced in order to mimick that in physical systems the infrared energy cascade terminates at the largest possible scale. The vorticity ω, with $\omega = \epsilon_{ij}\partial_i u_j$, is transported by the fluid and satisfies: $\partial_t\omega + (u\cdot\nabla)\omega - \nu\nabla^2\omega + \omega/\tau = F$ with $F = \epsilon_{ij}\partial_i f_j$ [6]. We choose the force to be gaussian, white-noice in time, with zero mean and two point function: $\langle f^j(x,t)\ f^k(y,s)\rangle = C^{jk}(x-y)\ \delta(t-s)$ where $C^{jk}(x)$, with $\nabla^j\, C^{jk}(x) = 0$, is a smooth function varying on a scale L_i and fastly decreasing at infinity. The correlation function of the vorticity forcing term is: $\langle F(x,t)\ F(y,s)\rangle = G(x-y)\ \delta(t-s)$ with $G = -\nabla^2\widehat{C}$. We shall assume translation, rotation and parity invariance and expand $C^{jk}(x)$ as: $C^{jk}(x) = \overline{\epsilon}\,\delta^{ij} - \overline{\epsilon}_w\,(3r^2\delta^{ij} - 2x^i x^j)/8 + \cdots$ with $r^2 = x^k x_k$. The scale L_i represents the injection length. The inverse cascade takes place at distances $L_i \ll x \ll L_f \simeq \tau^{3/2}\,\overline{\epsilon}^{1/2}$. At finite viscocity, there are two ultraviolet characteristic lengths, the usual dissipative length $l_d \simeq \nu^{1/2}\,\overline{\epsilon}_w^{-1/6}$ and another friction length $l_f \simeq \nu^{1/2}\,\tau^{1/2}$ above which friction dominates over dissipation. The direct cascade takes place at scale $l_f \ll x \ll L_i$. We shall consider the inviscid limit at fixed friction, ie. $\nu\to 0$ at τ fixed. Note that $(l_f/\,l_d)^2 = \tau\,\overline{\epsilon}_w^{1/3}$ is large in the quoted experiments.

The fundamental property of two dimensional turbulence recognized by Kraichnan [3] and Batchelor [8] is that the energy cascades towards the large scales because it cannot be dissipated at small scales. In absence of friction the system does not reach a stationnary state. But friction provides a way for the energy to escape and for the system to reach a stationnary state. The fact that the energy is not dissipated at small scales translates into the vanishing of the averaged energy dissipation rate, ie. $\nu\langle(\nabla u)^2\rangle = 0$ in the inviscid limit, which means that there are no energy dissipative anomaly [8], ie.

$$\lim_{\nu\to 0}\nu(\nabla u)^2_{(x)} = 0 \tag{2}$$

inside any correlation functions. However there could be enstrophy dissipative anomalies in the sense that

$\nu(\nabla\omega)^2$ does no vanish in the inviscid correlation functions:

$$\widehat{\epsilon}_w(x) \equiv \lim_{\nu\to 0} \nu(\nabla\omega)^2_{(x)} \tag{3}$$

Below we shall discuss possible consequences of the vanishing or not of $\widehat{\epsilon}_w$ in presence of friction.

Two and three point velocity correlation functions. In absence of energy dissipative anomaly, the mean energy density relaxes in the inviscid limit according to $\partial_t\langle\frac{u^2}{2}\rangle + \frac{1}{\tau}\langle u^2\rangle = \overline{\epsilon}$, showing that $\overline{\epsilon}$ is effectively the energy injection rate. It reaches a stationary limit with $\langle u^2\rangle = \overline{\epsilon}\,\tau$. Similarly the enstrophy density evolves according to $\partial_t\langle\omega^2\rangle + 2\nu\langle(\nabla\omega)^2\rangle + \frac{2}{\tau}\langle\omega^2\rangle = 2\overline{\epsilon}_w$ showing that $\overline{\epsilon}_w$ is the enstrophy injection rate. Let $\widehat{\epsilon}_w = \lim_{\nu\to 0}\nu\langle(\nabla\omega)^2\rangle$ be the enstrophy dissipation rate, $\widehat{\epsilon}_w < \overline{\epsilon}_w$. In the stationary limit at finite friction the enstrophy density is finite and equal to:

$$\langle\omega^2\rangle = \tau\,(\overline{\epsilon}_w - \widehat{\epsilon}_w) \tag{4}$$

Ie. the enstrophy density is equal to the difference of the enstrophy injection and the enstrophy dissipation rates times the friction relaxation time. The vorticity two point correlation function $\langle\omega(x)\omega(0)\rangle$ stay finite since it is bounded by $\langle\omega^2\rangle$. Hence the vorticity correlation cannot diverge as $x\to 0$, and it cannot have a negative anomalous dimension. Scalings $\langle\omega(x)\omega(0)\rangle \sim r^{-\xi'}$ or $\sim (\log r)^{\xi'}$ with $\xi' > 0$ are forbidden in presence of friction. At short distance one then may have:

$$\langle\omega(x)\omega(0)\rangle_{\nu=0} = \overline{\Omega} - (\tau\overline{\epsilon}_w)A\,(r/L)^{2\xi_2} + \cdots \tag{5}$$

with $\xi_2 > 0$. We shall denote the limiting value by $\overline{\Omega} = \tau(\overline{\epsilon}_w - \eta_w)$. By eq(9) below, η_w will be identified as the enstrophy transfer rate. The amplitude A, η_w and $\widehat{\epsilon}_w$, as well as the anomalous exponent ξ_2, are functions of the dimensionless parameter $\tau^3\overline{\epsilon}_w$.

Since $\nabla^2_x\langle(\delta u)^2\rangle = 2\langle\omega(x)\omega(0)\rangle$, finiteness of the vorticity two point function at coincidant points imply, $\langle(\delta u)^2)\rangle \simeq \frac{\overline{\Omega}}{2}\,r^2$, but the subleading terms could have anomalous scalings:

$$\langle(\delta u)^2\rangle = \frac{\overline{\Omega}}{2}\,r^2 - (\tau\overline{\epsilon}_w)\frac{A}{2(\xi_2+1)^2}r^2\,(r/L)^{2\xi_2} + \cdots \tag{6}$$

The leading scaling is the normal Kraichnan scaling but the amplitude is different, ie. $\overline{\Omega}$ instead of $\overline{\epsilon}_w^{2/3}$. There is a crossover at scale r_c, with $r_c^{2\xi_2} \sim (\overline{\Omega}/\tau\overline{\epsilon}_w)$, above which the second anomalous contribution dominates. It is interesting to compare the occurence of both the smooth r^2 term in eq.(6) and the anomalous contribution $r^{2\xi_2}$ in eq.(5) with the passive scalar problem with friction [6] in which anomalous zero modes occur in the passive scalar correlations only if the velocity flow is regular. The smooth r^2 term in $\langle(\delta u)^2\rangle$ does not contribute to the asymptotic large k behavior of the energy spectrum which thus scales as $k^{-3-2\xi_2}$. Namely:

$$E(k) \simeq -\frac{2^{2\xi_2+1}\Gamma(1+\xi_2)}{\Gamma(-\xi_2)}\,(\tau\overline{\epsilon}_w)A\,k^{-3}\,(kL)^{-2\xi_2} \tag{7}$$

as $k\to\infty$. The exponent ξ_2 is expected to be universal but not the amplitude A.

As usual, stationarity of the two point correlation functions gives in the inviscid limit, for $x \neq 0$:

$$\nabla^k_x\langle(\delta u^k)\,(\delta u)^2\rangle + 2\langle(\delta u)^2\rangle/\tau \quad = \quad 2(2\overline{\epsilon} - \widehat{C}(x)) \simeq \overline{\epsilon}_w\,r^2 + \cdots \tag{8}$$

It allows us to determine exactly the three point velocity inviscid structure functions:

$$\begin{aligned}
\langle(\delta u)^3_{\parallel}\rangle &= \frac{\eta_w}{8}r^3 + \frac{3A\overline{\epsilon}_w}{4(\xi_2+1)^2(\xi_2+2)(\xi_2+3)}\,r^{3+2\xi_2} + \cdots \\
\langle(\delta u)_{\parallel}(\delta u)^2_{\perp}\rangle &= \frac{\eta_w}{8}r^3 + \frac{3(2\xi_2+3)A\overline{\epsilon}_w}{4(\xi_2+1)^2(\xi_2+2)(\xi_2+3)}\,r^{3+2\xi_2} + \cdots
\end{aligned} \tag{9}$$

Without dissipative anomaly at finite friction. If there is no enstrophy dissipative anomalies in presence of friction, $\widehat{\epsilon}_w = 0$ and hence $\eta_w = 0$ assuming continuity of the correlations. The three point velocity structure (9) has an anomalous scaling at leading order. It behaves as

$$\langle(\delta u)^3\rangle \sim r^{3+2\xi_2}$$

since the first term $\eta_w r^3$ vanishes. As a consequence the enstrophy transfer is not constant through the scales, ie. there is no enstrophy cascade. The anomalous scaling manifests itself only at a subleading order in the velocity two point function but the enstrophy structure function is anomalous:

$$\langle(\delta\omega)^2\rangle = 2(\tau\overline{\epsilon}_w)A(r/L)^{2\xi_2}$$

Moreover in absence of enstrophy dissipative anomaly the one point vorticity correlations are gaussian,

$$\langle\exp(s\omega)\rangle = \exp(s^2(\tau\overline{\epsilon}_w)/2) \tag{10}$$

This is a direct consequence of the stationarity of the vorticity correlations. It is not true for the higher point functions. It could provide a way to check whether the enstrophy dissipation vanishes or not in presence of friction. Below we shall present a perturbative argument in favor of the absence of enstrophy dissipative anomalies in presence of friction.

In this scenario with $\widehat{\epsilon}_w = 0$, the quasi symmetry of the probability distribution may be explained by the fact that the even order correlations are parametrically larger that the odd order correlations, as it is for the two and three point function, cf eqs.(6,9) with $\eta_w = 0$. Namely the even order correlation have leading normal but anomalous subleading contributions whereas the odd order correlations would have anomalous contributions at leading order which decrease faster at small scales.

A scenario for the frictionless limit. Experimentally, $(\tau^3\overline{\epsilon}_w)$ is large and we are thus interested in the frictionless limit. We shall discuss this limit assuming the absence of dissipative anomaly at τ finite. Demanding that the three point structure function reproduces the known frictionless exact result with $\langle(\delta u)^3\rangle \sim r^3$ as $\tau \to \infty$ imposes that the anomalous exponent ξ_2 vanishes and that the amplitude A goes to one as $\tau \to \infty$. Moreover, demanding that the velocity structure function is finite as $\tau \to \infty$ requires:

$$\xi_2 = \zeta_2\,(\tau^3\overline{\epsilon}_w)^{-1/3} + o((\tau^3\overline{\epsilon}_w)^{-1/3}) \quad \text{and} \quad A = 1 + c\,(\tau^3\overline{\epsilon}_w)^{-1/3} + o((\tau^3\overline{\epsilon}_w)^{-1/3}) \tag{11}$$

with ζ_2 and c constant. Since ξ_2 is supposed to be universal but not the amplitude A, we expect ζ_2 universal but not c. As a consequence, at fixed non zero distance r, the vorticity correlation behaves logarithmically as:

$$\langle\omega(x)\omega(0)\rangle \simeq \overline{\epsilon}_w^{2/3}\ [\,c - 2\zeta_2\log(r/L)\,] + o((\tau^3\overline{\epsilon}_w)^{-1/3}) \tag{12}$$

Here we assume that the terms represented by the dots in eq.(5) which are subleading, and thus negligeable, at finite friction do not become predominant in the frictionless limit. But this is the simplest scenario.

The vorticity structure function, estimated as $\langle(\delta\omega)^2\rangle = 2\tau\overline{\epsilon}_w - 2\overline{\epsilon}_w^{2/3}\,(c - 2\zeta_2\log(r/L))$, is then dominated by the constant term since the enstrophy density diverges as $\tau \to \infty$. This may explain the experimental fact that vorticity correlations are almost constant in the inertial range. This scenario is compatible with experimental data since experimentally $\langle(\delta\omega)^2\rangle \simeq 10\,s^{-2}$ whereas $2\tau\overline{\epsilon}_w \simeq 20\,s^{-2}$ with an estimated error of $20\,^0/_0$ to $30\,^0/_0$. The velocity structure function will then be:

$$\langle(\delta u)^2\rangle \simeq \overline{\epsilon}_w^{2/3}\,r^2\ [c/2 + \zeta_2 - \zeta_2\log(r/L) + \cdots]$$

The smooth r^2 is not universal but the logarithmic correction is (since ζ_2 is expected to be universal). In the same way the energy spectrum (7) $E(k)$ becomes proportional to k^{-3} with:

$$E(k) \simeq 2\zeta_2\,\overline{\epsilon}_w^{2/3}\,k^{-3} \tag{13}$$

with Kraichnain's constant $C_K = 2\zeta_2$ which is universal if the anomalous dimensions at τ finite are universal. Note that the smooth but non universal r^2 terms in the velocity correlation does not contribute to the energy spectrum at large momenta. Of course, there could be logarithmic correction to the spectrum if $\zeta_2 = 0$.

With dissipative anomaly at finite friction. If the enstrophy dissipation is non zero in presence of friction, the enstrophy density is finite but smaller than the injected enstrophy density: $\langle\omega^2\rangle < \tau\overline{\epsilon}_w$. The three point velocity structure function then scales as $\langle(\delta u)^3\rangle \sim \eta_w r^3$ which allows us to identify η_w as the enstrophy transfer rate, ie. there is enstrophy cascade with transfer rate η_w. The quasi symmetry of the probability distribution function would hold only if the entrophy transfer rate is much smaller that the entrophy injection rate $\eta_w \ll \overline{\epsilon}_w$. There are many scenario for the frictionless limit depending on how $\eta_w/\overline{\epsilon}_w$, which is a function of $\tau^3\overline{\epsilon}_w$, behaves as $\tau \to \infty$.

Neglecting odd order correlations. We now discuss what would be the consequences of neglecting the odd order correlations in front of the even order ones. This is suggested by the quasi symmetry of the experimentally measured probability distribution. Consider the equations encoding the stationarity of an even number of velocity differences, ie. $\partial_t\langle(\delta u^{i_1})\cdots(\delta u^{i_{2n}})\rangle = 0$. Both the terms arising from the advection $(u\cdot\nabla)u$ or from the pressure ∇p involve correlation functions with an odd number of velocity insertions. Assuming that the odd order velocity correlations are much smaller that the even order ones, these terms are much smaller and negligeable compared to the remaining terms which are those arising from the friction and from the forcing and which involve an even number of velocity insertions. Hence at the leading order the stationarity condition of the even order correlations reads:

$$\frac{2n}{\tau}\langle(\delta u^{i_1})\cdots(\delta u^{i_{2n}})\rangle^{(0)} = 2\sum_{p<q}(\delta C^{i_p i_q}_{(x)})\,\langle\cdots\widehat{(\delta u^{i_p})}\cdots\widehat{(\delta u^{i_q})}\cdots\rangle^{(0)} \tag{14}$$

with $\delta C^{ij}(x) = C^{ij}(0) - C^{ij}(x)$. The overhatted quantities are omitted in the correlation functions. In eqs.(14) we have used the fact that there is no energy dissipative anomalies. The solution of eqs.(14) is provided by a gaussian statistics with zero mean and two-point function $\langle(\delta u^i)(\delta u^j)\rangle^{(0)} \simeq \frac{\tau\,\overline{\epsilon}_w}{8}(3r^2\delta^{ij} - 2x^i x^j)$. Hence, in this approximation the velocity structure functions scale as:

$$F^{(0)}_{2n} \equiv \langle(\delta u)^{2n}\rangle^{(0)} \quad\simeq\quad \text{const.}\ (\tau\overline{\epsilon}_w r^2)^n \tag{15}$$

This is compatible with eq.(6) since $\overline{\Omega} = \tau\overline{\epsilon}_w$ if $\eta_w \ll \overline{\epsilon}_w$. To be consistent it also has to solve the stationarity equations for the odd order correlations which at this order read:

$$\nabla^k_{x_1}\langle u^k_{(x_1)}u^{i_1}_{(x_1)}\,u^{i_2}_{(x_2)}\cdots u^{i_{2n-1}}_{(x_{2n-1})}\rangle^{(0)} + \nabla^{i_1}_{x_1}\langle p_{(x_1)}\,u^{i_2}_{(x_2)}\cdots u^{i_{2n-1}}_{(x_{2n-1})}\rangle^{(0)} + \text{perm.} = O(r^{2n}) \tag{16}$$

with $\nabla^2 p = -\nabla^j\nabla^k u^j u^k$. Since the statistics is gaussian it is enough to check them for $n = 2$. The gradiant of the pressure has scaling dimension one. Then the l.h.s. up to $O(r^4)$ is a rank three symmetric $O(2)$ polynomial tensor, transverse along all coordinates, invariant by translation and of scaling dimension three. There is no such tensor and thus the l.h.s. is $O(r^4)$ for $n = 2$. In other words, the n-point velocity correlations computed using the gaussian statistics are zero modes.

In view of the exact formula eqs.(6,9), this approximation would be better for the transverse velocity correlations. Note also that using this approximation to compute the one point vorticity functions $\langle\omega^n\rangle$ gives the exact result (10) if there is no enstrophy dissipative anomaly. Of course there would be deviations to this approximation:

$$F_n \simeq F^{(0)}_n + \delta F_n$$

with corrections δF_n including both contributions with normal scaling or with possible anomalous scaling as in eq.(6,9). In the scenario without enstrophy dissipation, these anomalous contributions will be subleading in the even order correlation but dominant at small scales in the odd order correlations. On contrary if the enstrophy transfer is not vanishing these corrections include contributions with normal scaling, since $\delta F_2 \simeq \eta_w\tau r^2$ and $\delta F_3 \simeq \eta_w r^3$ as follows from eqs.(6,9). Stationnarity of the three point functions leads then to an equation for the four point correlations of the form $\nabla^\pi F_4 + \frac{1}{\tau}F_3 = 0$ with ∇^π the transverse gradiant. Using $F_3 \simeq \eta_w r^3$ and naive scaling, ie. omitting possible zero mode contributions, gives a correction $\delta F_4 \simeq \frac{\eta_w}{\tau}r^4$ to the first order gaussian contribution $F^{(0)}_4$ which satisfies $\nabla^\pi F^{(0)}_4 = 0$. Similarly, stationarity of the four point functions now give equations of the form $\nabla^\pi F_5 + \frac{1}{\tau}\delta F_4 \simeq \eta_w\tau\overline{\epsilon}_w r^4$. For $\tau^3\overline{\epsilon}_w \gg 1$ as in [5], one may neglect $\frac{1}{\tau}\delta F_4$ in front of $\eta_w\tau\overline{\epsilon}_w r^4$, so that $\delta F_5 \simeq \eta_w\tau\overline{\epsilon}_w r^5$. Recursively, naive dimensional analysis apply to stationarity equations with $\tau^3\overline{\epsilon}_w \gg 1$ yield to first order in η_w: $\delta F_{n+2} \simeq \eta_w\,\tau^{1-n}\,(\tau^3\overline{\epsilon}_w)^k\,r^{n+2}$ with k the integer part of $n/3$. Of course $\delta F_{2n} \ll F^{(0)}_{2n}$ are small for $\tau^3\overline{\epsilon}_w \gg 1 \geq (\eta_w/\overline{\epsilon}_w)$. But zero modes of the stationarity equations which have been neglected in this dimensional analysis may give additional contributions to δF_n. These zero modes will be the leading terms for the odd order correlations if the enstrophy dissipation and thus the enstrophy transfer rate vanish.

MSR formalism. We now discuss how these scenario could fit in a field theory approach to the problem. The corrections (11) of the anomalous exponant as a function of the friction coefficient have a natural interpretation in the MSR formalism [10]. The MSR formalism provides a way to compute the inviscid vorticity correlations using a path integral with action

$$S = i\int dtdx\,\varphi_{(x,t)}(\partial_t\omega + u\cdot\nabla\omega + \frac{1}{\tau}\omega)_{(x,t)} + \frac{1}{2}\int dtdxdy\,\varphi_{(x,t)}G_{(x-y)}\varphi_{(y,t)} \tag{17}$$

Note that this action describes the inviscid limit $\nu = 0$.

Let us first define $\omega(x,t) = \overline{\epsilon}_w^{1/3}\widetilde{\omega}(t\overline{\epsilon}_w^{1/3}, x)$ and $\varphi(x,t) = \overline{\epsilon}_w^{-1/3}\widetilde{\varphi}(t\overline{\epsilon}_w^{1/3}, x)$. This corresponds to select configurations for which the dominant time scales and amplitudes are $\overline{\epsilon}_w^{-1/3}$. It allows us to present the theory as a perturbation of the frictionless action S_0:

$$S = S_0 + ig^{-1/3}\int dtdx\, \widetilde{\varphi}_{(x,t)}\widetilde{\omega}_{(x,t)} \tag{18}$$

with $g = \tau^3\overline{\epsilon}_w$ as dimensionless expansion parameter. This shows that the pertubation around the frictionless theory is controled by $(\tau^3\overline{\epsilon}_w)^{-1/3}$ as in eq.(12). Of course implementing this perturbation remains a challenge as the unperturbed theory is unkown.

Alternatively let us define $\omega(x,t) = (\tau\overline{\epsilon}_w)^{1/2}\omega'(t/\tau, x)$ and $\varphi(x,t) = (\tau\overline{\epsilon}_w)^{-1/2}\varphi'(t/\tau, x)$. This corresponds to select configurations with time scale τ. The theory is then formulated as a perturbation of the gaussian action by the advection term with $g^{1/2} = (\tau^3\overline{\epsilon}_w)^{1/2}$ as coupling constant:

$$S = S_{\rm gauss} + ig^{1/2}\int dtdx\, \varphi'_{(x,t)}(u'_{(x,t)}\cdot\nabla)\omega'_{(x,t)} \tag{19}$$

where $S_{\rm gauss}$ is the gaussian action obtained by neglecting the advection term. This is the statistics we found by neglecting odd order correlation functions. In the unperturbed theory $\langle\omega^2\rangle = \tau\overline{\epsilon}_w$ and there is no anomaly. Perturbation theory is then computable. We checked that there is no occurence of the dissipative anomaly in first order (one loop) in the perturbation theory. This is linked to the fact that the friction regularizes the theory in the inviscid limit such that there is no divergence in the Feymann amplitudes, at least in the first order we checked. This is expected to be true to all orders. In other words, there is no pertubative contribution to the enstrophy transfer rate, ie $\widehat{\epsilon}_w = \eta_w \simeq 0$ in perturbation theory.

Non perturbative contributions may arise due to instanton contributions to the path integral. Let us for exemple define $\omega(x,t) = \tau^{-1}\widehat{\omega}(t/\tau, x)$ and $\varphi(x,t) = (\tau^2\overline{\epsilon}_w)^{-1}\widehat{\varphi}(t/\tau, x)$. It allows us to write the action as $S = (\tau^3\overline{\epsilon}_w)^{-1}\widehat{S}$ with $\widehat{S}$ dimensionless. In the saddle point approximation, instantons could then potentially give a non perturbative contribution to the enstrophy transfer rate with $\eta_w \sim (\text{const.})\tau^{-3}\exp(-\text{const.}/(\tau^3\overline{\epsilon}_w))$.

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